On Proper Time in General Relativity

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It is shown how the relation $ds = c\,d\tau$ between the proper distance s and the proper time τ is obtained in general relativity. A general relation in curved spacetime between $d\tau$ and dt is given. This relation reduces to the special relativistic one for flat spacetime.

Time is not absolute; it elapses differently in different reference frames. Consider a reference frame S-the laboratory frame-in which the observer is at rest, and a second frame $S\prime$ which is translating in some arbitrary direction relative to S. The invariant interval between two events of a particle in flat spacetime is given by

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2 \tag{1}$$

in S, and by

$$ds\prime^2 = c^2dt\prime^2 - dx\prime^2 - dy\prime^2 - dz\prime^2 \tag{2}$$

in $S\prime$. These equations can also be written as

$$ds^2 = \left(1 - \frac{dx^2 + dy^2 + dz^2}{c^2dt^2}\right) c^2dt^2 = \left(1 - \frac{v^2}{c^2}\right) c^2dt^2, \tag{3}$$

and

$$ds\prime^2 = \left(1 - \frac{dx\prime^2 + dy\prime^2 + dz\prime^2}{c^2dt\prime^2}\right) c^2dt\prime^2 = \left(1 - \frac{v\prime^2}{c^2}\right) c^2dt\prime^2, \tag{4}$$

where v is the velocity of $S\prime$ relative to S and $v\prime$ is the velocity of the particle relative to $S\prime$. Now assume that the particle is at rest in $S\prime$, and thus both the particle and $S\prime$ move with velocity v relative to S. The proper time interval $d\tau$ between two events of the particle is defined by $d\tau = dt\prime$ where $dt\prime$ is the reading of time interval in the system $S\prime$ in which the particle is at rest, i.e. $v\prime = 0$. Then, using $ds = ds\prime$ we get

$$ds = c\,d\tau. \tag{5}$$

Equations (3) and (5) give

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}dt, \tag{6}$$

τ is the time registered by a fixed clock in $S\prime$ moving with the particle. Note that in a flat spacetime the clock in $S\prime$ need not be at the particle's position. Eq. (5) shows that proper time is an invariant quantity. What we have just presented is common knowledge in special relativity where the spacetime metric is the flat metric of Eq. (1). We now consider a gravitational field in a reference frame S with coordinates (ct, x, y, z) where the line element is described by

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu. \tag{7}$$

Let us consider another frame $S\prime$ with coordinates $(ct\prime, x\prime, y\prime, z\prime)$ moving relative to S. The line element in this frame is

$$ds\prime^2 = g_{\mu\nu}dx\prime^\mu dx\prime^\nu. \tag{8}$$

We shall assume that the form for ds^2 in Eq. (7) has already been diagonalized where the diagonal coordinates are the spherical coordinates $(c\,t, r, \theta, \phi)$ with the signature being $(+, -, -, -)$.

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It is our purpose in this work to show how the definition of proper time leads to Eq. (5) in general relativity too. Equation (7) can be written as

$$ds^2 = \left[1 - \frac{-\frac{g_{rr}}{g_{tt}}(\frac{dr}{dt})^2 - \frac{g_{\theta\theta}}{g_{tt}}(\frac{d\theta}{dt})^2 - \frac{g_{\phi\phi}}{g_{tt}}(\frac{d\phi}{dt})^2}{c^2}\right] c^2 g_{tt} dt^2. \tag{9}$$

Comparing this equation with Eq. (3) suggests that the velocity-squared in curved spacetime is defined by

$$v^2 = -\frac{g_{rr}}{g_{tt}}\left(\frac{dr}{dt}\right)^2 - \frac{g_{\theta\theta}}{g_{tt}}\left(\frac{d\theta}{dt}\right)^2 - \frac{g_{\phi\phi}}{g_{tt}}\left(\frac{d\phi}{dt}\right)^2. \tag{10}$$

We then find that Eq. (9) takes the special relativistic form

$$ds^2 = \left(1 - \frac{v^2}{c^2}\right) g_{tt} c^2 dt^2, \tag{11}$$

Here dt is the interval of coordinate time measured by a clock at the origin of the system S. Considering an event in $S\prime$ moving relative to $S\prime$ with velocity $v\prime$ we can write

$$ds\prime^2 = \left(1 - \frac{v\prime^2}{c^2}\right) g_{t\prime t\prime} c^2 dt\prime^2, \tag{12}$$

where $dt\prime$ is the interval of coordinate time between two neighboring events moving in $S\prime$ with velocity $v\prime$ as measured by a clock at the origin of $S\prime$.

As we have already stated before, the physical meaning of proper time is that it is the time measured by a clock at the position of the event which may be moving relative to S but at rest relative to $S\prime$. Thus putting $v\prime = 0$ $dt\prime$ becomes $d\tau$ and using $ds = ds\prime$ we get

$$ds = \sqrt{g_{t\prime t\prime}}\, c\, d\tau, \tag{13}$$

which is the general relation between ds and $d\tau$ for curved spacetime. Now, $g_{t\prime t\prime}$ can always chosen to be 1 and hence Eq. (13) reduces to Eq. (5) as promised. Equations (11) and (13) give

$$d\tau = \sqrt{1 - \frac{v^2}{c^2}}\, \sqrt{g_{tt}} dt, \tag{14}$$

which also has the special relativistic form and is the generalization of the flat spacetime equation, Eq. (6), to curved spacetime.

As an example, consider the frequently quoted system rotating about the z axis of the inertial frame described by

$$ds\prime^2 = c^2 dt\prime^2 - (dr\prime^2 + r^2 d\phi\prime^2 + dz\prime^2). \tag{15}$$

The transformation to the rotating system is given by

$$\begin{aligned} dt &= dt\prime, \\ d\phi &= d\phi\prime - \omega dt\prime, \\ dz &= dz\prime, \qquad dr = dr\prime, \end{aligned} \tag{16}$$

where ω is the angular velocity of the system. Taking for convenience the origins of the two systems to coincide, the line element in this rotating system is

$$ds^2 = (1 - \frac{\omega^2 r^2}{c^2}) c^2 dt^2 - (dr^2 + r^2 d\phi^2 + 2\omega r^2 d\phi dt + dz^2). \tag{17}$$

Thus $g_{tt} = (1 - \omega^2 r^2/c^2)$, $g_{t\prime t\prime} = 1$ and

$$d\tau = \left(1 - \omega^2 r^2/c^2\right)^{1/2} dt^2, \tag{18}$$

which follows from Eq. (14) with $v = 0$.